The Lee Fields Medal VIII: SOLUTIONS

1. In 1837, Dirichlet proved the following famous theorem about prime numbers:

   Dirichlet's Theorem. *Let a and d be positive whole numbers with no common factor greater than 1. Then there are infinitely many prime numbers of the form*

   $$a + nd \quad ; \qquad (\text{where } n \text{ is a natural number}).$$

   Use Dirichlet's Theorem to prove that there are infinitely many primes p such that both $p-2$ and $p+2$ are composite.

   Solution: This was statistically the hardest question on the paper with a mean average grade of 1.9. Arguably what was missing from students' efforts was some trial and error. For example, how about $a = 2$ and $d = 3$. There are infinitely many primes of the form:

   $$2 + 3n.$$

   However, while $2 + 3n - 2 = 3n$ is composite (for $n > 1$), we can't do much with $2+3n+2 = 4+3n$. We need both $a+nd+2$ and $a+nd-2$ to both have factorisations. We can play around but one possibility could be that $d = (a-2)(a+2)$ in which case:

   $$a + nd - 2 = (a-2) + (a-2)(a+2)n = (a-2)(1 + (a+2)n),$$

   and similarly $a+nd+2 = a+2+(a-2)(a+2)n = (a+2)(1+(a-2)n)$, both composite if $a - 2 > 1 \implies a > 3$. It cannot be that a is even (why?), so we try $a = 5$ and this gives $d = 3 \times 7 = 21$. There are infinitely many prime numbers of the form

   $$5 + 21n,$$

   and $5+21n-2 = 3+21n = 3(1+7n)$ and $5+21n+2 = 7+21n = 7(1+3n)$ are both composite. A 'smaller' solution is given by $7 + 15n$.

   This question was posed by Professor Emeritus Des MacHale of UCC, and appeared in the Bulletin of the Irish Mathematical Society (but without the hint of Dirichlet's Theorem).

2. Three non-zero complex numbers z_1, z_2, z_3 are in geometric progression and satisfy

$$z_1 + z_2 + z_3 = 0.$$

By finding the possible value of the common ratio r, or otherwise, show that r has non-zero imaginary part.

Solution: If the complex numbers are in geometric progression, they are equal to a, ar, and ar^2, and so:

$$z_1 + z_2 + z_3 = a + ar + ar^2 = a(1 + r + r^2) = 0 \implies a = 0 \text{ or } 1 + r + r^2 = 0.$$

It is not the case that $a = 0$ (why?), and so it must be the case that the common ratio satisfies

$$r^2 + r + 1 = 0.$$

There are two solutions, both of which have non-zero imaginary part by investigating

$$b^2 - 4ac = 1^2 - 4(1)(1) = -3 < 0,$$

or by finding them directly.

3. Given that $0 < b < a$ and $a^2 + b^2 = 6ab$, what is the value of

$$\frac{a+b}{a-b}?$$

Solution: This is just a bit of a trick, using the fact that both $(a+b)^2$ and $(a-b)^2$ are related to $a^2 + b^2$:

$$\begin{aligned} X &:= \frac{a+b}{a-b} \\ \implies X^2 &= \left(\frac{a+b}{a-b}\right)^2 \\ &= \frac{(a+b)^2}{(a-b)^2} \\ &= \frac{a^2+b^2+2ab}{a^2+b^2-2ab} \\ &= \frac{6ab+2ab}{6ab-2ab} \\ &= \frac{8ab}{4ab} = \frac{8}{4} \cdot \frac{ab}{ab} \underset{ab\neq 0}{=} 2 \\ \underset{\surd}{\implies} X &= \pm\sqrt{2}, \end{aligned}$$

but $X > 0$ as $a > b$ and so $X = \sqrt{2}$.

This question was provided by Dr Julie Crowley.

4. Draw the graph of each of the following equations:

   - $xy = 0$,
   - $x^2 + y^2 = 0$,
   - $x^2 - y^2 = 0$,
   - $x^3 - y^3 = 0$.

   Solution: Also provided by Professor Des MacHale, this requires the understanding that the graph of an equation consists of solutions to the equation:

   $$\text{points } (x, y) \text{ on graph} \iff \text{solutions } (x, y) \text{ of the equation.}$$

   (a) Using the No Zero Divisors Theorem:

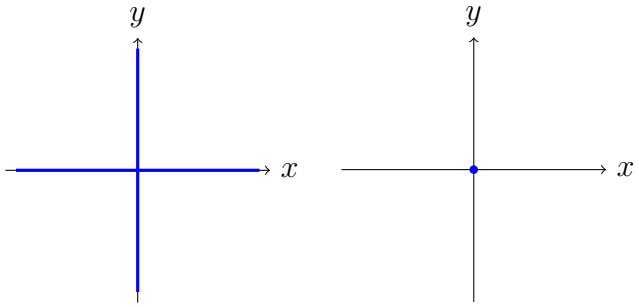
   $$xy = 0 \implies x = 0 \text{ or } y = 0,$$

   that is any point of the form $(0, y)$ or $(x, 0)$ is on the graph. That is, any point on the y- or x-axes.

   (b) Both x^2, $y^2 \geq 0$, and only equal to zero at zero. So $x^2 + y^2 > 0$ except for $x = 0$, $y = 0$. That is the only solution to the equation is $x = 0$, $y = 0$, corresponding to the origin.

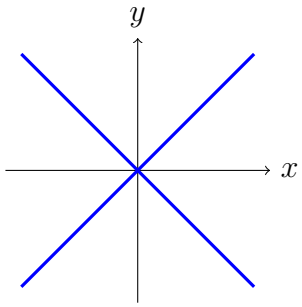
   (c) $x^2 - y^2 = 0 \implies (x - y)(x + y) = 0 \implies x - y = 0$ or $x + y = 0$. That is $y = x$ or $y = -x$. These are the equations of a pair of straight lines through the origin, one of slope one (45°), the other of slope -1.

   (d) $x^3 - y^3 = 0 \implies y^3 = x^3$. Unlike squaring ($\mathbb{R} \to \mathbb{R}$), cubing is invertible and so we can conclude $y = x$, that is we have a straight line graph through the origin.

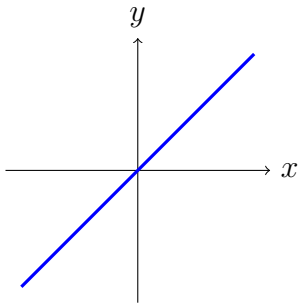


(a) $xy = 0$

(b) $x^2 + y^2 = 0$



(c) $x^2 - y^2 = 0$



(d) $x^3 - y^3 = 0$

5. How many different triangles with integer side lengths have perimeter 30?

   Solution: Also from Professor Des MacHale, let the side-lengths be a, b, , c. Let $0 < a \leq b \leq c$. The conditions are:

$$a + b + c = 30 \text{ and } a + b \geq c,$$

   the latter known as the *triangle inequality*. This means in particular, that $10 \leq c < 15$ (why?). Each gives a selection of solutions (c, a, b):

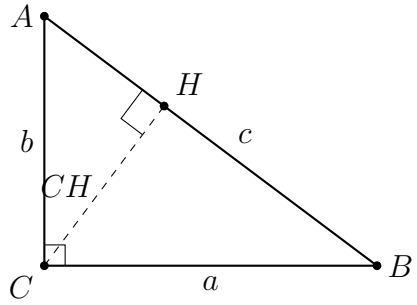
$$\begin{aligned}
&(10, 10, 10),\\
&(11, 11, 8),\ (11, 10, 9),\\
&(12, 12, 6),\ (12, 11, 7),\ (12, 10, 8),\ (12, 9, 9),\\
&(13, 13, 4),\ (13, 12, 5),\ (13, 11, 6),\ (13, 10, 7),\ (13, 9, 8),\\
&(14, 14, 2),\ (14, 13, 3),\ (14, 12, 4),\ (14, 11, 5),\ (14, 10, 6),\ (14, 9, 7),\ (14, 8, 8);
\end{aligned}$$

   that is 19 different triangles.

6. Let ABC be a triangle right-angled at C, and let CH be the perpendicular from the hypotenuse to C.

   Find the lengths of CH, AH, and BH in terms of the side lengths a, b, and c only.

   Solution: There are many, many different ways of answering this question; here is one. Here is a sketch of the triangle:



   We calculate the area using two different altitudes, that of b, and that of $|CH|$:

$$\begin{aligned}
\frac{1}{2}ab &= \frac{1}{2}c|CH|\\
\underset{\div_c \circ \times_2}{\Longrightarrow} |CH| &= \frac{ab}{c}
\end{aligned}$$

   Now we calculate $\cos(\angle ABC)$ in two different ways, in the smaller triangle, and in the larger one:

$$\begin{aligned}
\frac{|BH|}{a} &= \frac{a}{c}\\
\underset{\times a}{\Longrightarrow} |BH| &= \frac{a^2}{c}.
\end{aligned}$$

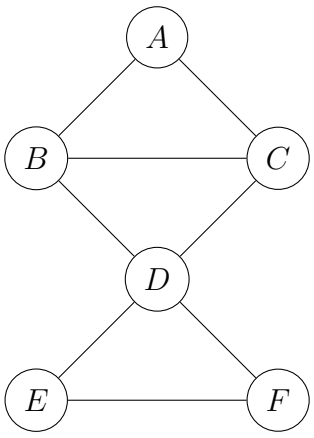
Similarly calculating $\cos(\angle BAC)$ gives:

$$|AH| = \frac{b^2}{c}.$$

7. Consider the following social network consisting of six individuals

$$A, B, C, D, E, F.$$

An edge between two vertices represents mutual friendship.



(a) What is the expected value of the number of friends of a randomly selected person?

(b) A person is selected uniformly at random, and then one of their friends is chosen uniformly at random. Find the expected number of friends of this friend.

(c) Which expected value is bigger?

Solution:

(a) The expected value of a random variable is given by:

$$\mathbb{E}[X] = \sum_{i=1}^{N} X_i \cdot \mathbb{P}[X = X_i].$$

If each value is equally likely, $\mathbb{P}[X = X_i] = 1/N$, then the expected value is just the (mean) average:

$$\mathbb{E}[X] = \sum_{i=1}^{N} X_i \cdot \frac{1}{N} = \frac{= \sum_{i=1}^{N} X_i}{N} = \overline{X}.$$

Let $X(P)$ be the number of friends of person P. The values are:

$$X(P) = 2,\ 3,\ 3,\ 4,\ 2,\ 2 \implies \overline{X} = \frac{8}{3} \approx 2.667.$$

(b) Let $Y(P)$ be the average number of friends of the friends of person P. So, for example, A is friends with B and C, who both have three friends, and so $Y(A) = 3$. In fact, all the $Y(P) = 3$, except D, with $Y(D) = 2.5$, and so:

$$\overline{Y} = \frac{5 \times 3 + 2.5}{6} = \frac{35}{12} \approx 2.917.$$

(c) The second expected value is larger, and indeed $\mathbb{E}[Y] \geq \mathbb{E}[X]$ in every social network where every person has at least one friend. Note the most popular person by definition sees friends with fewer friends than they have, and in the calculation of $\overline{Y}$, that person appears in many $Y(P)$, but only one $X(P)$. I think this is the basic reason for the phenomenon.

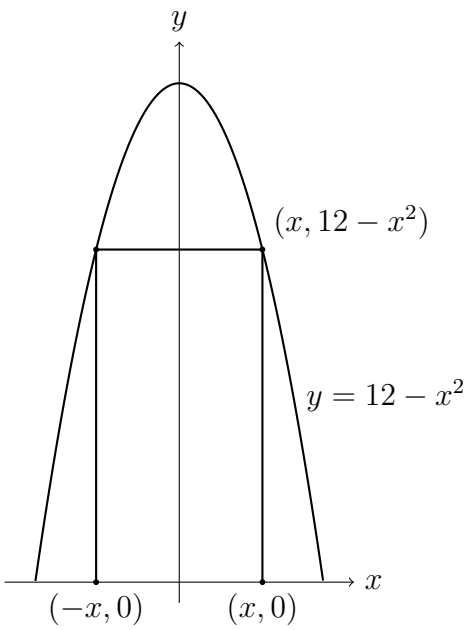
8. A rectangle has its lower two corners at $(-x, 0)$ and $(x, 0)$, and its upper two corners lie on the curve

$$y = 12 - x^2,$$

where $0 \leq x \leq \sqrt{12}$.

What is the greatest possible area of the rectangle?

Solution: Here is the situation:



The area of the rectangle is the base $b = 2x$ times the height $y = 12 - x^2$:

$$A(x) = x(12 - x^2) = 12x - x^3.$$

To find the location of the maximum we solve:

$$\begin{aligned} & \frac{dA}{dx} = 0 \\ \implies & 12 - 3x^2 = 0 \\ \underset{+3x^2}{\implies} & 3x^2 = 12 \\ & \underset{\div 3}{x^2} = 4 \\ \underset{\surd}{\implies} & x = \pm 2, \end{aligned}$$

but $x = -2$ is outside the interval $0 \leq x \leq \sqrt{12}$. The maximum area is therefore

$$A_{\max} = A(2) = 12(2) - (2)^3 = 24 - 8 = 16.$$

Note this is certainly a maximum, because at the interval end points $A(0) = A(\sqrt{12}) = 0$, and there are no other turning points.

9. Alice has eight aunts and uncles, where by an aunt or uncle we mean a sibling of one of Alice's parents. Each of these aunts and uncles has at most three children.

   (i) Suppose exactly k of Alice's aunts and uncles have three children. Show that Alice can have at most
   $$16 + k$$
   first cousins.

   (ii) Suppose Alice has 20 first cousins. Prove that at least four of her eight aunts and uncles must have three children.

Solution:

(a) Alice's aunts and uncles can have zero, one, two, or three children. If k have three children, then to maximise the number of first cousins that Alice has, $8 - k$ of them must have two children. Therefore the maximum number of first cousins Alice has is:

$$k \cdot 3 + (8 - k) \cdot 2 = 16 + k.$$

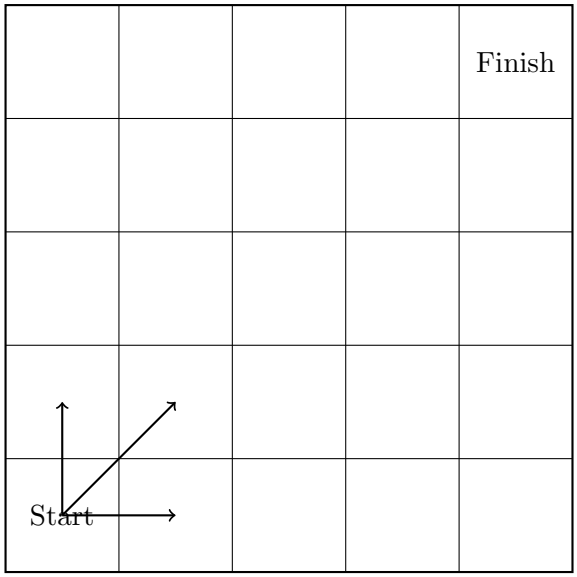
(b) Recall that Alice has a maximum of $16 + k$ first cousins, where k is the number of aunts/uncles that have three children. Therefore 20 must be less than or equal to this:

$$\begin{aligned} & 20 \leq 16 + k \\ \underset{-16}{\implies} & 4 \leq k \\ \implies & k \geq 4, \end{aligned}$$

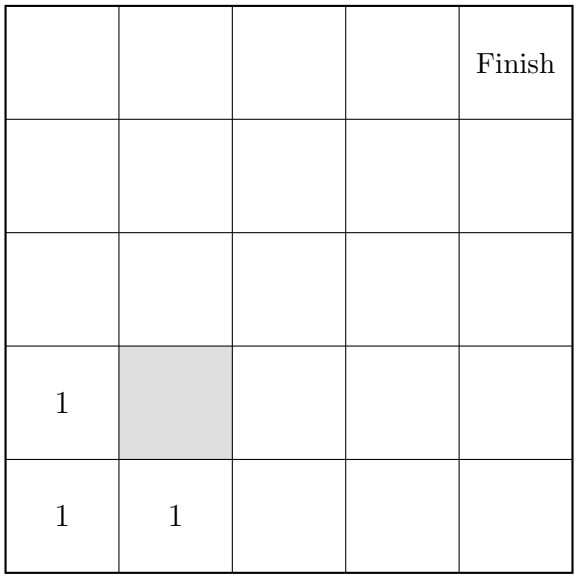
that is at least four of her aunts/uncles have at least three children.

10. On a 5×5 chessboard, how many ways are there for the King to move from the lower-left square to the top-right square?

    The only permitted moves are to adjacent squares moving up, right, and diagonally up and to the right.



Solution: The King has only one way to start, in the bottom left. Let us start by counting the number of ways for the King to go up, and to the right:



Now, how many ways can the King get to the shaded square? Well, up-right, right-up, and diagonal. This is exactly the sum of the three squares to left, down, and diagonally down left. Also, how many ways to keep going up, and keep going right: one way:

1				Finish
1				
1				
1	3			
1	1	1	1	1

Now, just continue in this fashion, using the rule that the number of ways to get to a square is the sum of the ways coming from below, from the left, and from diagonally down left:

1	9	41	129	321
1	7	25	63	129
1	5	13	25	41
1	3	5	7	9
1	1	1	1	1

The answer is 321.